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THE FUNDAMENTAL GROUP

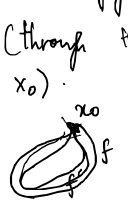
Def: Let X be a top. space, and $x_0 \in X$. A path in X that begins and ends at x_0 is called a "loop" based at x_0.

We define the "fundamental group of X relative to the base point x_0" as

$$\pi_1(X, x_0) = \{ [f] . \ f : [0,1] \longrightarrow X \text{ is a loop based at } x_0 \}$$

with group operation $[f] * [g] = [f * g]$.

Remark: The elements of $\pi_1(X, x_0)$ are path homotopy classes of loops based at x_0 (through the homotopy, all loops are based at x_0).



(ie, If F is a homotopy from f to f', then F_t is a loop based at x_0.)

Prop: $(\pi_1(X, x_0), *)$ is indeed a group.

$*$ is associative by the theorem proved previously.

identity element: $[e_{x_0}]$: $[f]*[e_{x_0}] = [e_{x_0}]*[f] = [f]$

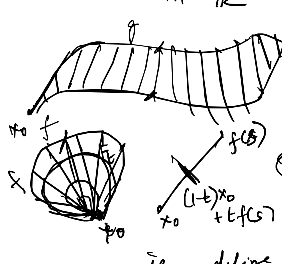
inverse: if f is a loop based at x_0, then $\bar{f}$ is a loop based at x_0. $\overline{[f]} = [\bar{f}]$.

Def: $[f]$ is said to be null-homotopic if $[f] = [e_{x_0}]$.

Example:

1. $\pi_1(\mathbb{R}^n, x_0)$ is the trivial group, ie, it has the identity element alone.

Proof: We showed previously that any two paths with the same starting and ending points in $\mathbb{R}^2$ are path homotopic. → (2)



A loop based at x_0 is a path starting and ending at x_0.

(2) ⇒ any loop f is homotopic to e_{x_0}.

ie, define $F : [0,1] \times [0,1] \longrightarrow \mathbb{R}^n$

$F_t(s) = (1-t)x_0 + tf(s)$.

2. If $X \subseteq \mathbb{R}^n$ is "convex", then $\pi_1(X, x_0)$ is trivial.

Proof: **Def**: A set $A \subseteq \mathbb{R}^n$ is said to be convex if $\forall a, b \in A$, the line segment from a to b lies in A.

ie, $\forall t \in [0,1]$, $(1-t)a + tb \in A$.

Given any loop $[f] \in \pi_1(X, x_0)$.
The straight line homotopy from f to e_{x_0} in $\mathbb{R}^n$ lies entirely inside X, since X is convex.

As a consequence, $\pi_1(B_n, x_0)$ is trivial for any $x_0 \in B_n$.

We'll show later that for $n \geq 3$, $\pi_1(S^{n-1}, x_0)$ is trivial.

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Group Homomorphisms

Def: A "homomorphism" $f : G \longrightarrow G'$ between groups G and G' is a map s.t.

$f(a*b) = f(a) * f(b) \ \forall a, b \in G$.

The "kernel" of f is $\{ a \in G \text{ s.t. } f(a) = e' \}$
this is denoted $\ker(f)$

Let $f : G \to G'$ be a homomorphism.

Prop: f satisfies $f(e) = e'$ and $f(\bar{a}) = \overline{f(a)}$ $\forall a, b \in G$.

Proof: Exercise.

Prop: $\ker(f)$ is a subgroup of G and $f(G)$ is a subgroup of G'.

Proof: If $a, b \in \ker(f)$, then $f(a*b) = f(a) * f(b) = e'$.
$\therefore a*b \in \ker(f)$.
$f(e) = e'$, so $e' \in \ker(f)$
If $a \in \ker(f)$ then $f(\bar{a}) = \overline{f(a)} = e'$.
$\therefore \bar{a} \in \ker(f)$.
$\therefore \ker(f)$ is a subgroup.
$f(G)$ is a subgroup – proved similarly, left as exercise.

Def: f is called an "isomorphism" if it is bijective.

(**Exercise**:) If $f : G \to G'$ is a bijective homomorphism then f^{-1} is a homomorphism.

Examples:

1. For any $a \in (\mathbb{R}, +)$

$$T_a : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto x + a$$

2. For any group $(G, *)$, $g \in G$,

$$T_g : G \longrightarrow G$$
$$a \longmapsto \bar{g} * a * g$$

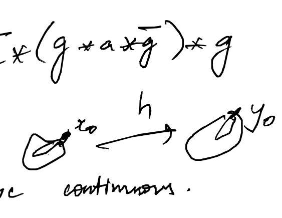
is a homomorphism.
(Conjugation)

Prop: The map T_g as defined above is an isomorphism.

Proof: $T_g(e) = g * e * \bar{g} = e$

$T_g^{-1}(a) = T_{\bar{g}}(a)$

$T_{\bar{g}} \circ T_g(a) = \bar{g} * (g * a * \bar{g}) * g$
$= a$



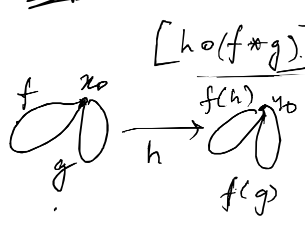
Def: Let $h : X \longrightarrow Y$ be continuous, and suppose $h(x_0) = y_0$.

Define $h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$
by $h_*([f]) = [h \circ f]$.

h_* is called the "homomorphism induced by h" relative to x_0.

Prop: h_* is a group homomorphism.

$[h \circ (f * g)] = [h \circ f] * [h \circ g]$



$[h \circ e_{x_0}] = [e_{y_0}]$.

Examples: 1. For $id : X \longrightarrow X$, $\forall x_0 \in X$, id_* is the identity map on $\pi_1(X, x_0)$

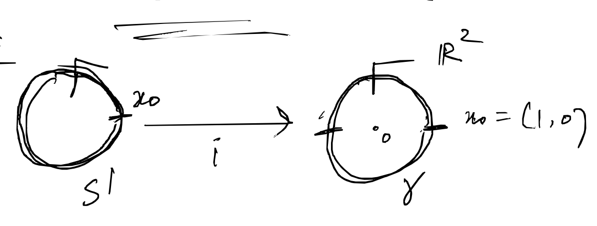
2. If $h : X \longrightarrow Y$ is $h(x) \equiv y_0 \ \forall x$
then $h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is the constant map

$h_*([f]) = [e_{y_0}]$.

3. The inclusion $i : S^1 \longrightarrow \mathbb{R}^2 \setminus \{0\}$ induces an isomorphism

$i_* : \pi_1(S^1, x_0) \longrightarrow \pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)$

Proof:



The map i_* is a group homomorphism.

It is also an inclusion : for this

We need to check $i_*([\gamma^n]) = i_*([\gamma^m])$
$\Rightarrow [\gamma^n] = [\gamma^m]$.

WTS:
$i_*([\gamma^{n-m}]) = [e_{x_0}] \Rightarrow [\gamma^{n-m}] = [e_{x_0}]$

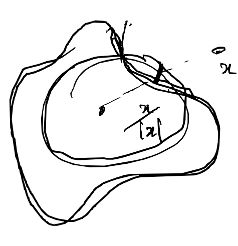
Unless $n - m = 0$, γ^{n-m} winds around the circle $n-m$ times but e_{x_0} doesn't.

Exercise: Winding number does not change under homotopy in $\mathbb{R}^2 \setminus \{0\}$.

Surjectivity: Need to show that any loop in $\mathbb{R}^2 \setminus \{0\}$ is obtained as $i^*[\gamma^n]$ for some n.

Consider the continuous function

$$F : \mathbb{R}^2 \setminus \{0\} \longrightarrow S^1$$
$$x \longmapsto \frac{x}{|x|},$$



$F \circ i = id_{S^1}$

$i \circ F|_{S^1} = id_{S^1}$

For any loop γ, I can isotope γ into a loop on S^1 through a homotopy.

Claim: This implies i_*, F_* are inverses of each other.

Claim follows due to the fact that

$F_* \circ i_* = id$ on $\pi_1(S^1, x_0)$
$i_* \circ F_* = id$ on $\pi_1(\mathbb{R}^2 \setminus 0, x_0)$

Corollary: $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)$ contains a non-trivial loop γ. So the upper and lower semi-circles that make up γ are not homotopic.

Rmk: If $h : X \to Y$ and $h(x_0) = y_0$, we will write $h : (X, x_0) \longrightarrow (Y, y_0)$.

Prop: $\forall \ h : (X, x_0) \longrightarrow (Y, y_0)$
$g : (Y, y_0) \longrightarrow (Z, z_0)$,

$(g \circ h)_* = g_* \circ h_*$

Proof:

$$(g \circ h)_*[\gamma] = [(g \circ h) \circ \gamma]$$
$$= g_*[h \circ \gamma]$$
$$= g_* \circ h_*[\gamma]$$